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ABSTRACT
We consider the first boundary value problem for a singularly perturbed parabolic PDE with convection on an interval. For the case of sufficiently smooth data, it is easy to construct a standard finite difference operator and a piecewise uniform mesh, condensing in the boundary layer, which gives an ε-uniformly convergent difference scheme. The order of convergence for such a scheme is exactly one and up to a small logarithmic factor one with respect to the time and space variables, respectively. In this paper we construct high-order time-accurate ε-uniformly convergent schemes by a defect correction technique. The efficiency of the new defect-correction scheme is confirmed by numerical experiments.

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1. Introduction

We consider the first boundary value problem for a singularly perturbed parabolic PDE with convection on an interval. The highest derivative in the equation is multiplied by an arbitrarily small parameter ε. When the parameter ε tends to zero, boundary layers may appear that give rise to difficulties when classical discretisation methods are applied, because the error in the approximate solution depends on the value of ε. An adapted placement of the nodes is needed to insure that the error is independent of the parameter value and depends only on the number of nodes in the mesh. Special schemes with this property are called ε-uniformly convergent. In the papers [1–5] we introduced and analyzed ε-uniformly convergent difference schemes for singularly perturbed boundary value problems for elliptic and parabolic equations. If the problem data are sufficiently smooth, for the parabolic equations with convection terms, the order of ε-uniform convergence for the scheme studied is exactly one and up to a small logarithmic factor one with respect to the time and space variables, respectively, i.e. $\mathcal{O}(N^{-1}\ln^2 N + K^{-1})$, where N and K are the number of intervals in the space and time discretisation. Because the amount of the computational work is proportional to the number K, higher order accuracy in time can considerably reduce computational expenses. Therefore, it is of interest to develop methods for which the order of convergence with respect to the time variable is increased.

For equations without convective terms the improvement of the accuracy in time, maintaining ε-uniform convergence by means of a defect correction technique, was also studied in [4, 5]. In this paper we develop schemes for which the order of convergence in time can be arbitrarily large if the solution is sufficiently smooth, similarly based on the defect correction principle, but now for a new class of singular perturbation problems, viz., for equations with convective terms. In contrast to our previous papers [4, 5], now we use a more elegant experimental technique for the determination of convergence orders, which we apply to convincingly verify the theoretical results.

2. The studied class of boundary value problems

On the domain $G = (0,1) \times (0,T]$, with boundary $S = \overline{G} \setminus G$ we consider the following singularly perturbed parabolic equation with Dirichlet boundary conditions[1]:

$$\begin{aligned} L_{(2.1)}u(x,t) &\equiv \left\{\varepsilon\, a(x,t)\frac{\partial^2}{\partial x^2} + b(x,t)\,\frac{\partial}{\partial x} - c(x,t) - p(x,t)\frac{\partial}{\partial t}\right\} u(x,t) = \\ &= f(x,t), \quad (x,t) \in G, \qquad (2.1a)\\ u(x,t) &= \varphi(x,t), \quad (x,t) \in S. \qquad (2.1b) \end{aligned}$$

For $S = S_0 \cup S^L$, we distinguish the lateral boundary $S^L = \{(x,t): \ x = 0 \text{ or } x = 1,\ 0 < t \le T\}$, and the initial boundary $S_0 = \{(x,t): \ x \in [0,1],\ t = 0\}$. In (2.1b) $a(x,t)$, $b(x,t)$, $c(x,t)$, $p(x,t)$, $f(x,t)$, $(x,t) \in \overline{G}$, and $\varphi(x,t)$, $(x,t) \in S$ are sufficiently smooth and bounded functions which satisfy

$$0 < a_0 \le a(x,t),\ 0 < b_0 \le b(x,t),\ 0 < p_0 \le p(x,t),\ c(x,t) \ge 0,\ (x,t) \in \overline{G}. \qquad (2.1c)$$

The real parameter ε from (2.1a) may take any values from half-open interval

$$\varepsilon \in (0,1]. \qquad (2.1d)$$

When the parameter ε tends to zero, the solution exhibits a layer in a neighbourhood of the set $S_1^L = \{(x,t): \ x = 0,\ 0 \le t \le T\}$, i.e., the left side of the lateral boundary. This layer is described by an ordinary differential equation (an ordinary boundary layer).

3. Difference scheme on an arbitrary mesh

To solve problem (2.1) we first consider a classical finite difference method. On the set $\overline{G}$ we introduce the rectangular mesh

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \qquad (3.1)$$

where $\overline{\omega}$ is the (possibly) non-uniform mesh of nodal points, x^i, in $[0,1]$, $\overline{\omega}_0$ is a uniform mesh on the interval $[0,T]$; N and K are the numbers of intervals in the meshes $\overline{\omega}$ and $\overline{\omega}_0$ respectively. We define $\tau = T/K$, $h^i = x^{i+1} - x^i$, $h = \max_i h^i$, $h \le M/N$, $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$.

Here and below we denote by M (or m) a sufficiently large (or small) positive constant which does not depend on the value of parameter ε or on the difference operators.

For problem (2.1) we use the difference scheme [9]

$$\Lambda_{(3.2)}z(x,t) = f(x,t), \qquad (x,t) \in G_h, \qquad (3.2a)$$

$$z(x,t) = \varphi(x,t), \qquad (x,t) \in S_h. \qquad (3.2b)$$

Here

[1] The notation is such that the operator $L_{(a.b)}$ is first introduced in equation $(a.b)$.

$$\begin{aligned}
\Lambda_{(3.2)} z(x,t) \equiv {} & \{\varepsilon\ a(x,t)\delta_{\overline{x}\widehat{x}} + b(x,t)\delta_x - c(x,t) - p(x,t)\delta_{\overline{t}}\}\, z(x,t), \\
\delta_{\overline{x}\widehat{x}} z(x^i,t) & = 2(h^{i-1}+h^i)^{-1}\left(\delta_x z(x^i,t) - \delta_{\overline{x}} z(x^i,t)\right), \\
\delta_{\overline{x}} z(x^i,t) & = (h^{i-1})^{-1}\left(z(x^i,t) - z(x^{i-1},t)\right), \\
\delta_x z(x^i,t) & = (h^i)^{-1}\left(z(x^{i+1},t) - z(x^i,t)\right), \\
\delta_{\overline{t}} z(x^i,t) & = \tau^{-1}\left(z(x^i,t) - z(x^i,t-\tau)\right),
\end{aligned}$$

$\delta_x z(x,t)$ and $\delta_{\overline{x}} z(x,t)$, $\delta_{\overline{t}} z(x,t)$ are the forward and backward differences, and the difference operator $\delta_{\overline{x}\widehat{x}} z(x,t)$ is an approximation of the operator $\frac{\partial^2}{\partial x^2}u(x,t))$ on the non-uniform mesh.

From [9] we know that the difference scheme (3.2), (3.1) is monotone. By means of the maximum principle and taking into account estimates of the derivatives (see Theorem 5 in the Appendix) we find that the solution of the difference scheme (3.2), (3.1) converges for a fixed value of the parameter ε as:

$$|\ u(x,t) - z(x,t)\ | \le M(\varepsilon^{-2}N^{-1} + \tau), \quad (x,t) \in \overline{G}_h. \tag{3.3}$$

This error bound for the classical difference scheme is clearly not ε-uniform.

The proof of (3.3) follows the lines of the classical convergence proof for monotone difference schemes (see [9, 10]). Taking into account the a-priori estimates for the solution, this results in the following theorem.

Theorem 1 *Assume in equation* (2.1) *that* $a,\ b,\ c,\ p,\ f \in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi \in H^{(\alpha+2n)}(\overline{G})$, $\alpha > 4$, $n = 0$ *and let the condition* (8.1) *with* $n = 0$ *be fulfilled. Then, for a fixed value of the parameter* ε, *the solution of* (3.2), (3.1) *converges to the solution of* (2.1) *with an error bound given by* (3.3).

4. The ε-uniformly convergent scheme

Here we discuss an ε-uniformly convergent method for (2.1) by taking a special mesh, condensed in the neighbourhood of the boundary layer. The location of the nodes is derived from a-priori estimates of the solution and its derivatives. The way to construct the mesh for problem (2.1) is the same as in [4, 5, 11]. More specifically, we take

$$\overline{G}_h^{\,*} = \overline{\omega}^*(\sigma) \times \overline{\omega}_0\ , \tag{4.1}$$

where $\overline{\omega}_0$ is the uniform mesh with step-size $\tau = T/K$, i.e. $\overline{\omega}_0 = \overline{\omega}_{0(3.1)}$, and $\overline{\omega}^* = \overline{\omega}^*(\sigma)$ is a special *piecewise* uniform mesh depending on the parameter $\sigma \in I\!R$, which depends on ε and N. We take $\sigma = \sigma_{(4.1)}(\varepsilon, N) = \min(\,1/2,\ m^{-1}\varepsilon \ln N\,)$, where $m = m_{(4.1)}$ is an arbitrary number with a size of about $m_0 = \min_{\overline{G}}[a^{-1}(x,t)b(x,t)]$. The mesh $\overline{\omega}^*(\sigma)$ is constructed as follows. The interval $[\,0,1\,]$ is divided in two parts $[\,0,\sigma\,]$, $[\,\sigma,1\,]$, $\sigma \le 1/2$. In each part we use a uniform mesh, with $N/2$ subintervals in $[\,0,\sigma\,]$ and $[\,\sigma,1\,]$.

Theorem 2 *Let the conditions of Theorem* 1 *hold. Then the solution of* (3.2), (4.1) *converges* ε-*uniformly to the solution of* (2.1) *and the following estimate holds:*

$$|u(x,t) - z(x,t)| \le M(N^{-1}\ln N + \tau), \quad (x,t) \in \overline{G}_h^*. \tag{4.2}$$

The proof of this theorem can be found in [10, 12].

5. Numerical results for scheme (3.2), (4.1)

To see the effect of the special mesh in practice, we take the model problem

$$L_{(5.1)}u(x,t) \equiv \left\{\varepsilon\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right\}u(x,t) = f(x,t), \quad (x,t) \in G, \tag{5.1}$$

$$u(x,t) = \varphi(x,t), \quad (x,t) \in S,$$

where

$$f(x,t) = -4t^3, \quad (x,t) \in \overline{G}, \quad \varphi(x,t) = 0, \quad (x,t) \in S; \quad T = 1.$$

For the approximation of problem (5.1) we use the scheme (3.2), (4.1), where $m = 2^{-1}$, $\overline{G}_h = \overline{G}_h^*$.

Since the exact solution for this problem is unknown, we replace it by the numerical solution U_ε^{2048} computed on a finest mesh $\overline{G}_h$ with $N = K = 2048$, for each value of ε. Then the computed maximum pointwise error is defined by

$$E(N,K,\varepsilon) = \max_{(x,t)\in\overline{G}_h} |\, z(x,t) - u^*(x,t)\,|. \tag{5.2}$$

Here $u^*(x,t)$ is the linear interpolation obtained from the reference solution U_ε^{2048} corresponding to the numerical solution $z(x,t)$ of problem (3.2), (4.1). We compute $E(N,K,\varepsilon)$ for various values of ε, N, K. Note that no special interpolation is needed along the t-axis.

The results are given in Table 1. From the analysis of the numerical results we conclude that, in agreement with (4.2), the order of convergence for large $N = K$ is $\mathcal{O}(N^{-1}\ln N + K^{-1})$, i.e. almost one with respect to the space and time variables. Thus, it corresponds with the theoretical results.

$\varepsilon \setminus N$	16	32	64	128	256	512
1.0	2.178-03	1.179-03	6.046-04	2.989-04	1.410-04	6.073-05
2^{-1}	6.460-03	3.555-03	1.840-03	9.126-04	4.312-04	1.859-04
2^{-2}	1.533-02	8.465-03	4.402-03	2.188-03	1.035-03	4.467-04
2^{-3}	2.950-02	1.639-02	8.544-03	4.257-03	2.017-03	8.708-04
2^{-4}	4.819-02	3.275-02	2.238-02	1.148-02	5.510-03	2.399-03
2^{-5}	6.342-02	3.601-02	2.334-02	1.454-02	8.192-03	4.061-03
2^{-6}	7.341-02	4.263-02	2.409-02	1.498-02	8.460-03	4.192-03
2^{-7}	7.763-02	4.651-02	2.495-02	1.521-02	8.601-03	4.269-03
2^{-8}	7.939-02	4.819-02	2.618-02	1.534-02	8.669-03	4.307-03
2^{-9}	8.015-02	4.893-02	2.673-02	1.540-02	8.707-03	4.326-03
2^{-10}	8.050-02	4.927-02	2.699-02	1.543-02	8.728-03	4.336-03
2^{-18}	8.082-02	4.984-02	2.730-02	1.547-02	8.749-03	4.345-03
$\overline{\mathbf{E}}$**(N)**	**8.082-02**	**4.984-02**	**2.730-02**	**1.547-02**	**8.749-03**	**4.345-03**

Table 1: Errors $\boldsymbol{E(N,N,\varepsilon)}$ for model problem (5.1) with the special scheme (3.2), (4.1)
In this table the function $\boldsymbol{E(N,K,\varepsilon)}$ is defined by (5.2). The bottom line, $\overline{\boldsymbol{E}}\boldsymbol{(N)}$, gives the computed maximum for each column.

6. Improved time-accuracy

6.1 A scheme based on defect correction

In this section we construct a discrete method based on defect correction, which also converges ε-uniformly to the solution of the boundary value problem, but with an order of accuracy (with respect to τ) higher than in (4.2).

The technique used in this paper to improve time-accuracy is based on the one in [4, 5]. For the difference scheme (3.2), (4.1) the error in the approximation of the partial derivative $(\partial/\partial t)\, u(x,t)$ is caused by the divided difference $\delta_{\overline{t}}\, z(x,t)$ and is associated with the truncation error given by the relation

$$\frac{\partial u}{\partial t}(x,t) - \delta_{\overline{t}}\, u(x,t) = 2^{-1}\,\tau\frac{\partial^2 u}{\partial t^2}(x,t) - 6^{-1}\,\tau^2\frac{\partial^3 u}{\partial t^3}(x,t-\vartheta), \quad \vartheta \in [0,\tau]. \tag{6.1}$$

Therefore, we now use for the approximation of $(\partial/\partial t)\, u(x,t)$ the expression

$$\delta_{\overline{t}}\, u(x,t) + \tau\delta_{\overline{t}\,\overline{t}}\, u(x,t)/2,$$

where $\delta_{\bar{t}\bar{t}}\, u(x,t) \equiv \delta_{t\,\bar{t}}\, u(x, t-\tau)$. Notice that $\delta_{t\,\bar{t}}\, u(x,t)$ is the second central divided difference. We can evaluate a better approximation than (3.2a) by defect correction

$$\Lambda_{(3.2)} z^c(x,t) = f(x,t) + 2^{-1} p(x,t) \tau \, \frac{\partial^2 u}{\partial t^2}(x,t), \tag{6.2}$$

with $x \in \overline{\omega}$ and $t \in \overline{\omega}_0$, where $\overline{\omega}$ and $\overline{\omega}_0$ are as in (3.1); τ is step-size of the mesh $\overline{\omega}_0$; $z^c(x,t)$ is the "corrected" solution. Instead of $(\partial^2/\partial t^2)\, u(x,t)$ we shall use $\delta_{\bar{t}\,\bar{t}}\, z(x,t)$, where $z(x,t)$, $(x,t) \in G_{h(4.1)}$ is the solution of the difference scheme (3.2), (4.1). We may expect that the new solution $z^c(x,t)$ has a consistency error $\mathcal{O}(\tau^2)$. This is true, as will be shown in Section 6.3.

Moreover, in a similar way we can construct a difference approximation with a convergence order higher than two (with respect to the time variable) and $\mathcal{O}(N^{-1} \ln N)$ with respect to the space variable ε-uniformly.

6.2 The defect correction scheme of second-order accuracy in time

We denote by $\delta_{k\bar{t}} z(x,t)$ the backward difference of order k:

$$\delta_{k\bar{t}}\; z(x,t) = \left(\delta_{k-1\,\bar{t}}\; z(x,t) - \delta_{k-1\,\bar{t}}\; z(x,t-\tau)\right)/\tau, \quad t \ge k\tau, \quad k \ge 1;$$
$$\delta_{0\bar{t}}\; z(x,t) = z(x,t), \quad (x,t) \in \overline{G}_h.$$

To construct the difference schemes of second order accuracy in τ in (6.2), instead of $(\partial^2/\partial t^2) u(x,t)$ we use $\delta_{2\,\bar{t}}\, z(x,t)$, the second divided difference of the solution to the discrete problem (3.2), (4.1). On the mesh $\overline{G}_h$ we write the finite difference scheme (3.2) in the form

$$\begin{aligned} \Lambda_{(3.2)} z^{(1)}(x,t) &= f(x,t), \quad (x,t) \in G_h, \\ z^{(1)}(x,t) &= \varphi(x,t), \quad (x,t) \in S_h, \end{aligned} \tag{6.3}$$

where $z^{(1)}(x,t)$ is the uncorrected solution. For the corrected solution $z^{(2)}(x,t)$ we solve the problem for $(x,t) \in G_h$

$$\Lambda_{(3.2)} z^{(2)}(x,t) = f(x,t) + \left\{ \begin{array}{ll} p(x,t) 2^{-1}\, \tau \dfrac{\partial^2}{\partial t^2} u(x,0), & t = \tau, \\ p(x,t) 2^{-1}\, \tau \delta_{2\,\bar{t}}\, z^{(1)}(x,t), & t \ge 2\tau \end{array} \right\}, \quad (x,t) \in G_h,$$
$$z^{(2)}(x,t) = \varphi(x,t), \quad (x,t) \in S_h. \tag{6.4}$$

Here the derivative $\dfrac{\partial^2 u}{\partial t^2}(x,0)$ is obtained from the equation (2.1a). We shall call $z^{(2)}(x,t)$ the solution of difference scheme (6.4), (6.3), (4.1) (or shortly, (6.4), (4.1)).

For simplicity, in the remainder of this section we suppose that the coefficients $a(x,t)$, $b(x,t)$ do not depend on t

$$a(x,t) = a(x), \quad b(x,t) = b(x), \quad (x,t) \in \overline{G} \tag{6.5}$$

and we take a homogeneous initial condition:

$$\varphi(x,0) = 0, \quad x \in [0,1]. \tag{6.6}$$

Under the conditions (6.5), (6.6), the following estimate holds for the solution of problem (6.4), (4.1)

$$\left| u(x,t) - z^{(2)}(x,t) \right| \le M \left[\, N^{-1} \ln N + \tau^2 \,\right], \quad (x,t) \in \overline{G}_h. \tag{6.7}$$

Theorem 3 *Let conditions* (6.5), (6.6) *hold and assume in equation* (2.1) *that* $a,\ b,\ c,\ p,\ f \in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi \in H^{(\alpha+2n)}(\overline{G})$, $\alpha > 4$, $n = 1$ *and let condition* (8.1) *be satisfied for* $n = 1$. *Then for the solution of difference scheme* (6.4), (4.1) *the estimate* (6.7) *holds.*

Proof. The proof of Theorem 3 is given in the Appendix, see Section 8.2.

6.3 The defect correction scheme of third-order accuracy in time

The above procedure can be used to obtain an arbitrarily large order of accuracy in time. Here we only show how to construct the difference scheme of third order accuracy. On the grid $\overline{G}_h$ we consider the difference scheme

$$\Lambda_{(3.2)}\, z^{(3)}(x,t) \;=\; f(x,t) \;+ \tag{6.8a}$$

$$+\left\{\begin{array}{ll} p(x,t)\left(C_{11}\tau\dfrac{\partial^2}{\partial t^2}u(x,0)+C_{12}\tau^2\dfrac{\partial^3}{\partial t^3}u(x,0)\right), & t=\tau,\\ p(x,t)\left(C_{21}\tau\dfrac{\partial^2}{\partial t^2}u(x,0)+C_{22}\tau^2\dfrac{\partial^3}{\partial t^3}u(x,0)\right), & t=2\tau,\\ p(x,t)\left(C_{31}\tau\delta_{2\,\overline{t}}z^{(2)}(x,t)+C_{32}\tau^2\delta_{3\,\overline{t}}z^{(1)}(x,t)\right), & t\ge 3\tau \end{array}\right\},\quad (x,t)\in G_h,$$

$$z^{(3)}(x,t) \;=\; \varphi(x,t), \qquad (x,t)\in S_h.$$

Here $z^{(1)}(x,t)$ and $z^{(2)}(x,t)$ are the solutions of problems (6.3), (4.1) and (6.4), (4.1) respectively, the derivatives $(\partial^2/\partial t^2)u(x,0)$, $(\partial^3/\partial t^3)u(x,0)$ are again obtained from equation (2.1a). The coefficients C_{ij} are chosen such that they satisfy the following conditions

$$\frac{\partial}{\partial t}u(x,t)=\delta_{\overline{t}}u(x,t)+C_{11}\tau\frac{\partial^2}{\partial t^2}u(x,t-\ \tau)+C_{12}\tau^2\frac{\partial^3}{\partial t^3}u(x,t-\ \tau)+\mathcal{O}(\tau^3),$$

$$\frac{\partial}{\partial t}u(x,t)=\delta_{\overline{t}}u(x,t)+C_{21}\tau\frac{\partial^2}{\partial t^2}u(x,t-2\tau)+C_{22}\tau^2\frac{\partial^3}{\partial t^3}u(x,t-2\tau)+\mathcal{O}(\tau^3),$$

$$\frac{\partial}{\partial t}u(x,t)=\delta_{\overline{t}}u(x,t)+C_{31}\tau\delta_{2\,\overline{t}}u(x,t)+C_{32}\tau^2\delta_{3\,\overline{t}}u(x,t)+\mathcal{O}(\tau^3).$$

It follows that

$$C_{11}=C_{21}=C_{31}=1/2,\quad C_{12}=C_{32}=1/3,\quad C_{22}=5/6. \tag{6.8b}$$

By $z^{(3)}(x,t)$ we denote the solution of the difference scheme (6.8), (4.1) and again, for simplicity, we assume the homogeneous initial condition

$$\varphi(x,0)=0,\quad f(x,0)=0,\quad x\in[0,1]\,. \tag{6.9}$$

Under conditions (6.5), (6.9) the following estimate holds for the solution of difference scheme (6.8), (4.1)

$$\left|\,u(x,t)-z^{(3)}(x,t)\,\right|\le M\;\left[\;N^{-1}\ln\,N+\tau^3\;\right],\quad (x,t)\in\overline{G}_h. \tag{6.10}$$

Theorem 4 *Let conditions* (6.9) *hold and assume in equation* (2.1) *that* $a,\ b,\ c,\ p,\ f\in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi\in H^{(\alpha+2n)}(\overline{G})$, $\alpha>4$, $n=2$ *and let condition* (8.1) *be satisfied with* $n=2$. *Then for the solution of scheme* (6.8), (4.1) *the estimate* (6.10) *is valid.*

Proof. The proof of Theorem 4 is given in the Appendix in Section 8.3.

In a similar way we could construct difference schemes with an arbitrary high order of accuracy

$$\mathcal{O}(N^{-1}\ \ln\,N+\tau^{n+1}),\quad n>2.$$

7. Numerical results for the time-accurate schemes

We find the solution of the following boundary value problem

$$L_{(5.1)}u(x,t)=0,\quad 0<x<1,\quad 0<t\le T,\quad T=1. \tag{7.1}$$

$$u(0,t)=t^4,\quad 0<t\le T,\quad u(x,t)=0,\quad (x,t)\in S,\quad x>0.$$

It should be noted that the solution of this problem is singular.

It is very attractively to use the analytical solution of problem (7.1) for the computation of the errors in the approximate solution, as was in [4, 5]. But here a suitable representation (for computation) of the solution $u(x,t)$ is unknown. Instead of the exact solution, it is possible to use the solution of the discrete problem on a very fine mesh. But this method is not effective because the analysis of the order of accuracy for a defect-correction scheme requires a very dense mesh that leads not only to large computational expenses but also to large round-off errors.

Hence, we use the method from [6], different from the above-mentioned techniques. The solution of problem (7.1) is represented in the form of a sum

$$u(x,t) = V^{(1)}(x,t) + v(x,t), \quad (x,t) \in \overline{G}, \tag{7.2}$$

where $V^{(1)}(x,t)$ is the main singular part (two first terms) of the asymptotic expansion of the solution of problem (7.1), and $v(x,t)$ is the remainder term, which is a sufficiently small smooth function. The function $V^{(1)}(x,t)$ has a sufficiently simple analytical representation

$$V^{(1)}(x,t) = V_0(x,t) + V_1(x,t), \quad (x,t) \in \overline{G}, \qquad \text{where}$$

$$V_0(x,t) = t^4\Psi(x), \quad \Psi(x) = [\exp(-\varepsilon^{-1}x) - \exp(-\varepsilon^{-1})]/[1 - \exp(-\varepsilon^{-1})],$$

$$V_1(x,t) = -4\,t^3\,x\,\exp(-x/\varepsilon)\,/[1 - \exp(-1/\varepsilon)],$$

$$|V_0(x,t)| \le M, \quad |V_1(x,t)| \le M\varepsilon, \quad (x,t) \in \overline{G}.$$

The function $v(x,t)$ is the solution of the problem

$$L_{(5.1)}v(x,t) = f_0(x,t), \quad (x,t) \in G, \tag{7.3}$$

$$v(0,t) = 0, \quad v(1,t) = -V_1(1,t), \quad 0 < t \le T, \qquad v(x,0) = 0, \quad 0 < x < 1,$$

with

$$f_0(x,t) = -4\,t^2\,[t\,\exp(-1/\varepsilon) + 3\,x\,\exp(-x/\varepsilon)]\,/\,[1 - \exp(-1/\varepsilon)]\,.$$

For the function $v(x,t)$ the following estimate holds:

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} v(x,t)\right| \le M\varepsilon^2[1 + \varepsilon^{-k}], \quad (x,t) \in \overline{G}, \quad k + 2k_0 \le 4, \quad k \le 3.$$

Then the function $v(x,t)$ and the product $\varepsilon^2(\partial^4/\partial x^4)v(x,t)$ are ε-uniformly bounded. Thus, we can consider $v(x,t)$ as the regular part of this solution.

(1.) We solve the discrete problem, which approximates the boundary value problem (7.3), on the finest available mesh $\overline{G}_h = \overline{G}_{(*)}h(4.1)$ for $N = K = 2048$, for the chosen value of ε, and there are no difficulties to find the function $v(x,t) = v_\varepsilon^{2048}(x,t)$ and the reference solution

$$u(x,t)_{(7.2)} = u_\varepsilon^{2048}(x,t) = V^{(1)}(x,t) + v_\varepsilon^{2048}(x,t).$$

(2.) Further for solving problem (7.1), we –successively– use the scheme (6.3), (4.1) and the defect correction schemes (6.4), (4.1) and (6.8), (4.1) to find the functions $z^{(1)}(x,t)$, $z^{(2)}(x,t)$ and $z^{(3)}(x,t)$, respectively. Note that $z^{(1)}(x,t)$ is the uncorrected solution, $z^{(2)}(x,t)$ and $z^{(3)}(x,t)$ are the corrected solutions. In these cases we compute the maximum pointwise errors $E(N,K,\varepsilon)$ by formula (5.2), where $u^*(x,t)$ is the linear interpolation obtained from the reference solution $u_\varepsilon^{2048}(x,t)$ corresponding with the numerical solution $z^{(k)}(x,t)$, $k = 1,2,3$ for the values $N = 2^i$, $i = 2,3,\ldots,10$, $K = 2^j$, $j = 2,3,\ldots,10$.

The computational process (1.) and (2.) is repeated for all values of $\varepsilon = 2^{-n}$, $n = 0,2,4,\ldots,12$. As a result, we get $E(N,K,\varepsilon)$ for various values of ε, N, K for each of the functions $z^{(1)}(x,t)$, $z^{(2)}(x,t)$,

$K \setminus N$	4	8	16	32	64	128	256	512	1024
					$z^{(1)}$				
4	**1.51-1**	1.42-1	1.03-1	6.86-2	4.39-2	2.67-2	1.58-2	9.21-3	5.36-3
8	1.39-1	**1.38-1**	1.02-1	6.82-2	4.36-2	2.65-2	1.56-2	9.00-3	5.15-3
16	1.32-1	1.36-1	**1.02-1**	6.80-2	4.35-2	2.63-2	1.55-2	8.88-3	5.03-3
32	1.29-1	1.35-1	1.01-1	**6.79-2**	4.34-2	2.63-2	1.54-2	8.81-3	4.97-3
64	1.27-1	1.35-1	1.01-1	6.79-2	**4.34-2**	2.62-2	1.54-2	8.78-3	4.93-3
128	1.26-1	1.35-1	1.01-1	6.78-2	4.34-2	**2.62-2**	1.53-2	8.76-3	4.92-3
256	1.26-1	1.35-1	1.01-1	6.78-2	4.34-2	2.62-2	**1.53-2**	8.75-3	4.91-3
512	1.26-1	1.34-1	1.01-1	6.78-2	4.34-2	2.62-2	1.53-2	**8.75-3**	4.91-3
1024	1.26-1	1.34-1	1.01-1	6.78-2	4.34-2	2.62-2	1.53-2	8.75-3	**4.90-3**
					$z^{(2)}$				
4	**1.33-1**	1.37-1	1.02-1	6.81-2	4.35-2	2.64-2	1.55-2	8.89-3	5.05-3
8	1.28-1	**1.35-1**	1.01-1	6.79-2	4.34-2	2.62-2	1.54-2	8.79-3	4.94-3
16	1.26-1	1.35-1	**1.01-1**	6.78-2	4.34-2	2.62-2	1.53-2	8.76-3	4.91-3
32	1.26-1	1.34-1	1.01-1	**6.78-2**	4.34-2	2.62-2	1.53-2	8.75-3	4.90-3
64	1.25-1	1.34-1	1.01-1	6.78-2	**4.34-2**	2.62-2	1.53-2	8.75-3	4.90-3
128	1.25-1	1.34-1	1.01-1	6.78-2	4.34-2	**2.62-2**	1.53-2	8.75-3	4.90-3
256	1.25-1	1.34-1	1.01-1	6.78-2	4.34-2	2.62-2	**1.53-2**	8.75-3	4.90-3
512	1.25-1	1.34-1	1.01-1	6.78-2	4.34-2	2.62-2	1.53-2	**8.75-3**	4.90-3
1024	1.25-1	1.34-1	1.01-1	6.78-2	4.34-2	2.62-2	1.53-2	8.75-3	**4.90-3**
					$z^{(3)}$				
4	**1.28-1**	1.35-1	1.01-1	6.79-2	4.34-2	2.62-2	1.54-2	8.78-3	4.94-3
8	1.26-1	**1.34-1**	1.01-1	6.78-2	4.34-2	2.62-2	1.53-2	8.75-3	4.91-3
16	1.25-1	1.34-1	**1.01-1**	6.78-2	4.34-2	2.62-2	1.53-2	8.75-3	4.90-3
32	1.25-1	1.34-1	1.01-1	**6.78-2**	4.34-2	2.62-2	1.53-2	8.75-3	4.90-3
64	1.25-1	1.34-1	1.01-1	6.78-2	**4.34-2**	2.62-2	1.53-2	8.75-3	4.90-3
128	1.25-1	1.34-1	1.01-1	6.78-2	4.34-2	**2.62-2**	1.53-2	8.75-3	4.90-3
256	1.25-1	1.34-1	1.01-1	6.78-2	4.34-2	2.62-2	**1.53-2**	8.75-3	4.90-3
512	1.25-1	1.34-1	1.01-1	6.78-2	4.34-2	2.62-2	1.53-2	**8.75-3**	4.90-3
1024	1.25-1	1.34-1	1.01-1	6.78-2	4.34-2	2.62-2	1.53-2	8.75-3	**4.90-3**

Table 2: Errors $\boldsymbol{E(N,K)}$ for $\boldsymbol{\varepsilon = 2^{-10}}$

$z^{(3)}(x,t)$. Analyzing these results, we observe convergence of the solutions for increasing $N = K$ for any of the functions $z^{(1)}$, $z^{(2)}$, $z^{(3)}$ and for all values of ε used. In order to show this result we give Table 2 only for $\varepsilon = 2^{-10}$. The corresponding tables for other values of ε are similar.

In Table 2 the values of $E(N,K)$ are given for the functions $z^{(1)}$, $z^{(2)}$ and $z^{(3)}$. For each of them we see decreasing errors for $N = K$, i.e. we have ε-uniform convergence. But the order of convergence, which we observe, is about equal to one for all functions. All errors corresponding to the same values of N, K but to different $z^{(k)}$ are similar.

We know that the error of approximation consists of two parts. One part is due to the discretisation of the space derivatives and the second is due to the time discretisation. We briefly call these components the space error and the time error. Since by the defect correction we improve only the accuracy with respect to the time, we expect a decreasing time error. It can be much smaller than the space error and therefore the observed error in Table 2 corresponds only to the space error. In order to show this we split the combined error in the space error (Table 3) and the time error (Table 5). The structure of Table 3 is similar to that of Table 2.
Table 3 contains the values of the space errors computed by the formula

$$E^{(s)}(N_i,K) = E(N_i,K) - E(N_{i+1},K), \quad i = 3,4,\ldots,9, \quad N_i = 2^i.$$

We see that the errors are the same for all different K. The order of the errors in Table 2 and Table 3 are the same. From Table 3 we deduce Table 4, where the ratios of the space errors is given by

$$R^{(s)}(N_i,K) = E^{(s)}(N_i,K)/E^{(s)}(N_{i+1},K), \quad i = 3,4,\ldots,8.$$

In Table 4 we see the first order of the convergence with respect to the space up to a small logarithmic

$K \setminus N$	8	16	32	64	128	256	512
				$z^{(1)}$			
4	3.85-2	3.47-2	2.47-2	1.72-2	1.09-2	6.59-3	3.85-3
8	3.61-2	3.40-2	2.46-2	1.72-2	1.09-2	6.59-3	3.85-3
16	3.48-2	3.36-2	2.45-2	1.72-2	1.09-2	6.59-3	3.85-3
32	3.41-2	3.34-2	2.45-2	1.72-2	1.09-2	6.58-3	3.85-3
64	3.38-2	3.32-2	2.45-2	1.72-2	1.09-2	6.58-3	3.85-3
128	3.36-2	3.32-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3
256	3.35-2	3.32-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3
512	3.35-2	3.31-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3
1024	3.35-2	3.31-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3
				$z^{(2)}$			
4	3.50-2	3.36-2	2.45-2	1.72-2	1.09-2	6.59-3	3.85-3
8	3.39-2	3.33-2	2.45-2	1.72-2	1.09-2	6.58-3	3.85-3
16	3.36-2	3.32-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3
32	3.35-2	3.31-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3
64	3.35-2	3.31-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3
128	3.34-2	3.31-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3
...	...	...	...	...	...	...	...
1024	3.34-2	3.31-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3
				$z^{(3)}$			
4	3.41-2	3.33-2	2.45-2	1.72-2	1.09-2	6.58-3	3.85-3
8	3.35-2	3.32-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3
16	3.35-2	3.31-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3
32	3.34-2	3.31-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3
...	...	...	...	...	...	...	...
1024	3.34-2	3.31-2	2.45-2	1.72-2	1.09-2	6.58-3	3.84-3

Table 3: Space Errors $\boldsymbol{E^{(s)}(N,K)}$ for $\boldsymbol{\varepsilon = 2^{-10}}$

factor.

In a similar way we construct Table 5 for the time error:

$$E^{(t)}(N,K_j) = E(N,K_j) - E(N,K_{j+1}), \quad j = 2,3,\dots,9$$

and Table 6 for their ratios

$$R^{(t)}(N,K_j) = E^{(t)}(N,K_j)/E^{(t)}(N,K_{j+1}), \quad j = 2,3,4,\dots,8, \quad K_j = 2^j.$$

In Table 5 we now observe a number of interesting results:

1. We see that the time error is essentially smaller than the space error. This explains the fact that we do not see the effect of the time error in Table 2.
2. The errors for $z^{(1)}$ are larger than those for $z^{(2)}$ and, similarly, the errors for $z^{(2)}$ are larger than those for $z^{(3)}$.
3. We see that approximately the same error ($\approx 10^{-6}$) is obtained for $z^{(1)}$ at $K = 512$, for $z^{(2)}$ at $K = 32$ and for $z^{(3)}$ at $K = 16$. Because the computational work is proportional to K, we see that the amount of work is essentially reduced by the defect correction.
4. Table 6, which shows the ratios of the errors, confirms the theoretical order of convergence as derived in Section 6. From the theory, the solution $\boldsymbol{z^{(1)}(x,t)}$ of problem (6.3), (4.1) converges with rate $\boldsymbol{\mathcal{O}(\tau)}$ (estimate (4.2) and Theorem 2). The solution $\boldsymbol{z^{(2)}(x,t)}$ of problem (6.4), (4.1), where $\boldsymbol{z^{(1)}(x,t)}$ is the solution of problem (6.3), (4.1), converges with $\boldsymbol{\mathcal{O}(\tau^2)}$ (estimate (6.7) and Theorem 3). The solution $\boldsymbol{z^{(3)}(x,t)}$ of problem (6.8), (4.1), where $\boldsymbol{z^{(2)}(x,t)}$ and $\boldsymbol{z^{(1)}(x,t)}$ are the solutions of problems (6.4), (4.1) and (6.3), (4.1) respectively, converges with $\boldsymbol{\mathcal{O}(\tau^3)}$ (estimate (6.10) and Theorem 4). The corresponding reduction factors are easily recognized in Table 6.

$K \setminus N$	8	16	32	64	128	256
			$\boldsymbol{z^{(1)}}$			
4	1.11	1.40	1.44	1.58	1.65	1.71
8	1.06	1.38	1.43	1.58	1.65	1.71
16	1.04	1.37	1.43	1.58	1.65	1.71
32	1.02	1.36	1.43	1.58	1.65	1.71
64	1.02	1.36	1.43	1.58	1.65	1.71
128	1.01	1.36	1.43	1.58	1.65	1.71
...	...	...	...	...	...	...
1024	1.01	1.36	1.43	1.58	1.65	1.71
			$\boldsymbol{z^{(2)}}$			
4	1.04	1.37	1.43	1.58	1.65	1.71
8	1.02	1.36	1.43	1.58	1.65	1.71
16	1.01	1.36	1.43	1.58	1.65	1.71
...	...	...	...	...	...	...
1024	1.01	1.35	1.43	1.58	1.65	1.71
			$\boldsymbol{z^{(3)}}$			
4	1.02	1.36	1.43	1.58	1.65	1.71
8	1.01	1.36	1.43	1.58	1.65	1.71
...	...	...	...	...	...	...
1024	1.01	1.35	1.43	1.58	1.65	1.71

Table 4: Ratios of Space Errors $\boldsymbol{R^{(s)}(N, K)}$ for $\boldsymbol{\varepsilon = 2^{-10}}$

Conclusion

In this paper we have shown a defect correction procedure that can be used to easily improve the time-accuracy for a parabolic singularly perturbed PDE, still obtaining ε-uniform first order accuracy in the space discretisation.

The error in the approximation consists of two parts: one part due to space discretisation and the other due to the time discretisation. The defect correction process only improves the accuracy with respect to the time and it does not change the error with respect to the space variable. Using a new experimental technique for the determination of the convergence orders, we have numerically revealed the time and space error components separately. For the meshes used, the time error has been found of the order 10^{-4}–10^{-11} whereas the space error is of the order 10^{-2}–10^{-3}. Thus, the time error is considerably smaller than the space error, and the total error is practically equal to the space error.

Numerical results confirm the theoretical bound that the order of convergence with respect to the space variable is close to one, uniform in ϵ. The numerical experiments also confirm that by defect correction we are able to increase the time accuracy of the approximate solution from 1st to 2nd and 3rd order. As a result, we can essentially decrease the number of time intervals and therefore the amount of computational work.

$K \setminus N$	4	8	16	32	64	128	256	512	1024
					$z^{(1)}$				
4	1.24-2	3.46-03	1.08-03	3.64-04	2.44-04	2.23-04	2.15-04	2.11-04	2.08-04
8	6.63-3	1.91-03	6.14-04	2.08-04	1.39-04	1.27-04	1.23-04	1.21-04	1.19-04
16	3.42-3	1.00-03	3.26-04	1.11-04	7.44-05	6.80-05	6.56-05	6.44-05	6.37-05
32	1.73-3	5.12-04	1.68-04	5.73-05	3.84-05	3.51-05	3.39-05	3.32-05	3.29-05
64	8.73-4	2.59-04	8.54-05	2.91-05	1.95-05	1.78-05	1.72-05	1.69-05	1.67-05
128	4.38-4	1.30-04	4.30-05	1.47-05	9.84-06	8.98-06	8.67-06	8.51-06	8.42-06
256	2.19-4	6.54-05	2.16-05	7.37-06	4.94-06	4.51-06	4.35-06	4.27-06	4.22-06
512	**1.10-4**	**3.27-05**	**1.08-05**	**3.69-06**	**2.47-06**	**2.26-06**	**2.18-06**	**2.14-06**	**2.12-06**
					$z^{(2)}$				
4	5.39-3	1.64-03	5.47-04	1.86-04	1.24-04	1.13-04	1.09-04	1.07-04	1.06-04
8	1.55-3	4.82-04	1.61-04	5.36-05	3.54-05	3.23-05	3.12-05	3.06-05	3.03-05
16	4.14-4	1.31-04	4.36-05	1.43-05	9.41-06	8.58-06	8.27-06	8.12-06	8.03-06
32	**1.07-4**	**3.40-05**	**1.13-05**	**3.69-06**	**2.42-06**	**2.21-06**	**2.13-06**	**2.09-06**	**2.07-06**
64	2.73-5	8.68-06	2.88-06	9.35-07	6.14-07	5.59-07	5.40-07	5.29-07	5.24-07
128	6.87-6	2.19-06	7.28-07	2.36-07	1.55-07	1.41-07	1.36-07	1.33-07	1.32-07
256	1.73-6	5.51-07	1.83-07	5.91-08	3.88-08	3.53-08	3.41-08	3.34-08	3.31-08
512	4.32-7	1.38-07	4.58-08	1.48-08	9.71-09	8.85-09	8.53-09	8.37-09	8.28-09
					$z^{(3)}$				
4	2.67-3	7.37-04	2.09-04	5.74-05	3.53-05	3.19-05	3.08-05	3.02-05	2.99-05
8	3.15-4	9.26-05	2.66-05	7.17-06	4.41-06	3.99-06	3.85-06	3.77-06	3.73-06
16	**4.13-5**	**1.23-05**	**3.45-06**	**9.00-07**	**5.51-07**	**4.99-07**	**4.81-07**	**4.72-07**	**4.67-07**
32	5.36-6	1.60-06	4.38-07	1.13-07	6.88-08	6.23-08	6.01-08	5.90-08	5.83-08
64	6.84-7	2.04-07	5.51-08	1.41-08	8.60-09	7.79-09	7.51-09	7.37-09	7.29-09
128	8.64-8	2.57-08	6.91-09	1.76-09	1.08-09	9.74-10	9.39-10	9.21-10	9.11-10
256	1.09-8	3.23-09	8.65-10	2.20-10	1.34-10	1.22-10	1.17-10	1.15-10	1.14-10
512	1.36-9	4.04-10	1.08-10	2.75-11	1.68-11	1.52-11	1.47-11	1.44-11	1.42-11

Table 5: Time Errors $\boldsymbol{E^{(t)}(N,K)}$ for $\boldsymbol{\varepsilon = 2^{-10}}$

$K \backslash N$	4	8	16	32	64	128	256	512	1024
					$z^{(1)}$				
4	1.87	1.81	1.77	1.75	1.75	1.75	1.75	1.75	1.75
8	1.94	1.91	1.88	1.87	1.87	1.87	1.87	1.87	1.87
16	1.97	1.95	1.94	1.94	1.94	1.94	1.94	1.94	1.94
32	1.99	1.98	1.97	1.97	1.97	1.97	1.97	1.97	1.97
64	1.99	1.99	1.98	1.98	1.98	1.98	1.98	1.98	1.98
128	2.00	1.99	1.99	1.99	1.99	1.99	1.99	1.99	1.99
256	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
					$z^{(2)}$				
4	3.49	3.40	3.39	3.47	3.50	3.51	3.51	3.51	3.51
8	3.73	3.69	3.70	3.75	3.77	3.77	3.77	3.77	3.77
16	3.86	3.84	3.85	3.88	3.89	3.89	3.89	3.89	3.89
32	3.93	3.92	3.93	3.94	3.94	3.94	3.94	3.94	3.94
64	3.97	3.96	3.96	3.97	3.97	3.97	3.97	3.97	3.97
128	3.98	3.98	3.98	3.99	3.99	3.99	3.99	3.99	3.99
256	3.99	3.99	3.99	3.99	3.99	3.99	3.99	3.99	3.99
					$z^{(3)}$				
4	8.49	7.96	7.85	8.00	8.01	8.00	8.00	8.00	8.00
8	7.62	7.52	7.72	7.97	8.00	8.00	8.00	8.00	8.00
16	7.71	7.71	7.87	7.99	8.00	8.00	8.00	8.00	8.00
32	7.84	7.85	7.94	8.00	8.00	8.00	8.00	8.00	8.00
64	7.92	7.92	7.97	8.00	8.00	8.00	8.00	8.00	8.00
128	7.96	7.96	7.99	8.00	8.00	8.00	8.00	8.00	8.00
256	7.98	7.98	7.99	8.00	8.00	8.00	8.00	8.00	8.00

Table 6: Ratios of Time Errors $\boldsymbol{R^{(t)}(N,K)}$ for $\boldsymbol{\varepsilon = 2^{-10}}$

References

1. *Farrell P.A., Hemker P.W., Shishkin G.I.* Discrete approximations for singularly perturbed boundary value problems with parabolic layers. In three parts: J. Comput. Mathematics 14 (1996) 71–97; J. Comput. Mathematics 14 (1996) 183–194; J. Comput. Mathematics 14 (1996) 273–290.
2. *Farrell P.A., Miller J.J.H., O'Riordan E. and Shishkin G.I.* A Uniformly Convergent Finite Difference Scheme for a Singularly Perturbed Semilinear Equation, SIAM Journal on Numerical Analysis 33 (1996) 1135–1149.
3. *Hemker P.W. and Shishkin G.I.* On a class of singularly perturbed boundary value problems for which an adaptive mesh technique is necessary. In: D.Bainov and V.Covachev, eds, Proceedings of the Second International Colloquium on Numerical Analysis, VSP, International Science Publishers, 1994. pp 83–92.
4. *Hemker P.W. , Shishkin G.I. and Shishkina L.P.* The use of defect correction for the solution of parabolic singular perturbation problems. ZAMM 77 (1997) 59–74.
5. *Hemker P.W. , Shishkin G.I. and Shishkina L.P.* $\mathcal{E}$-uniform schemes with high-order time-accuracy for parabolic singular perturbation problems. IMA J. Numer. Anal. 20 (2000) 99–121.
6. *Kolmogorov V.L. and Shishkin G.I.* Numerical methods for singularly perturbed boundary value problems modeling diffusion processes. In: Miller J.J.H., ed. Singular Perturbation Problems in Chemical Physics. New York: John Wiley & Sons, 1997. pp 181–362.
7. *Ladyzhenskaya O.A. and Ural'tseva N.N.* Linear and Quasi-linear Equations of Elliptic Type. M: Nauka, 1973 (in Russian).
8. *Miller, J. J. H., O'Riordan, E., Shishkin, G. I.* Fitted Numerical Methods for Singular Perturbation Problems. Singapore: World Scientific, 1996.
9. *Samarski A.A.* Theory of Difference Scheme. M: Nauka, 1989 (in Russian).
10. *Shishkin G.I.* Grid Approximation of Singularly Perturbed Elliptic and Parabolic Equations. Ekaterinburg: UrO RAN, 1992 (in Russian).
11. *Shishkin G.I.* Approximation of solutions for singularly perturbed boundary value problems with parabolic boundary layers. Zh. Vychisl. Mat. Mat. Fiz. 29 (1989) 963–977 (in Russian).
12. *Shishkin G.I.* Grid approximation of singularly perturbed elliptic equation in domain with characteristic bounds. Sov. J. Numer. Anal. Math. Modeling 5 (1990) 327–343.
13. *Shishkin G.I.* Approximation of solutions for singularly perturbed boundary value problems with corner boundary layers. Zh. Vychisl. Mat. Mat. Fiz. 27 (1987) 1360–1374 (in Russian).

Appendix

8.1 **Estimates of the solution and its derivatives**

In this Appendix we rely on the a-priori estimates for the solution of problem (2.1) on the domain $G = D \times [0,T]$, and its derivatives as derived for elliptic and parabolic equations in [13, 11, 10].

We denote by $H^{(\alpha)}(\overline{G}) = H^{\alpha,\alpha/2}(\overline{G})$ the Hölder space, where α is an arbitrary positive number [7]. We suppose that the functions $f(x,t)$ and $\varphi(x,t)$ satisfy compatibility conditions at the corner points, so that the solution of the boundary value problem is smooth for every fixed value of the parameter ε.

For simplicity, we assume that at the corner points $S_0 \cap \overline{S}_1$ the following conditions hold

$$\begin{aligned} &\frac{\partial^k}{\partial x^k}\varphi(x,t) = \frac{\partial^{k_0}}{\partial t^{k_0}}\varphi(x,t) = 0, \quad k + 2k_0 \le [\ \alpha\ ] + 2n, \\ &\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} f(x,t) = 0, \qquad\qquad k + 2k_0 \le [\ \alpha\ ] + 2n - 2, \end{aligned} \tag{8.1}$$

where $[\ \alpha\ ]$ is the integer part of a number α, $\alpha > 0$, $n \ge 0$ is an integer. We also suppose that $[\ \alpha\ ] + 2n \ge 2$.

Using interior a-priori estimates and estimates up to the boundary for the regular function $\widetilde{u}(\xi,t)$ (see [7]), where $\widetilde{u}(\xi,t) = u(x(\xi),t)$, $\xi = x/\varepsilon$, we find for $(x,t) \in \overline{G}$ the estimate

$$\left| \frac{\partial^{k+k_0}}{\partial x^k\, \partial t^{k_0}} u(x,t) \right| \le M\,\varepsilon^{-k}, \quad k + 2k_0 \le 2n+4, \quad n \ge 0. \tag{8.2}$$

This estimate holds, for example, for

$$u \in H^{(2n+4+\nu)}(\overline{G}), \quad \nu > 0, \tag{8.3}$$

where ν is some small number.

For example, (8.3) is guaranteed for the solution of (2.1) if the coefficients satisfy $a \in H^{(\alpha+2n-1)}(\overline{G})$, $c,\ p,\ f \in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi \in H^{(\alpha+2n)}(\overline{G})$, $\alpha > 4$, $n \ge 0$ and condition (8.1) is fulfilled.

In fact we need a more accurate estimate than (8.2). Therefore, we represent the solution of the boundary value problem (2.1) in the form of the sum

$$u(x,t) = U(x,t) + W(x,t), \quad (x,t) \in \overline{G}, \tag{8.4}$$

where $U(x,t)$ represents the regular part, and $W(x,t)$ the singular part, i.e. the parabolic boundary layer. The function $U(x,t)$ is the smooth solution of equation (2.1a) satisfying condition (2.1b) for $t = 0$. For example, under suitable assumptions for the data of the problem, we can consider the solution of the Dirichlet boundary value problem for equation (2.1a) smoothly extended to the domain $\overline{G}^*$ beyond of S_1^L ($\overline{G}^*$ is a sufficiently large neighbourhood of $\overline{G}$ beyond of S_1^L.) On the domain $\overline{G}$ the coefficients and the initial value of the extended problem are the same as for (2.1). Then the function $U(x,t)$ is the restriction (on $\overline{G}$) of the solution to the extended problem, and $U \in H^{(2n+4+\nu)}(\overline{G})$, $\nu > 0$. The function $W(x,t)$ is the solution of a boundary value problem for the parabolic equation

$$L_{(2.1)} W(x,t) = 0, \quad (x,t) \in G, \quad W(x,t) = u(x,t) - U(x,t), \quad (x,t) \in S. \tag{8.5}$$

If (8.3) is true then $W \in H^{(2n+4+\nu)}(\overline{G})$. Now, for the functions $U(x,t)$ and $W(x,t)$ we derive the estimates

$$\left| \frac{\partial^{k+k_0}}{\partial x^k\, \partial t^{k_0}} U(x,t) \right| \le M, \tag{8.6}$$

$$\left| \frac{\partial^{k+k_0}}{\partial x^k\, \partial t^{k_0}} W(x,t) \right| \le M\,\varepsilon^{-k} \exp(-m_{(8.7)}\varepsilon^{-1} r(x,\gamma)\,), \tag{8.7}$$

$$(x,t) \in \overline{G}, \quad k + 2k_0 \le 2n+2,$$

where $r(x,\gamma)$ is the distance between the point $x\in[0,1]$ and the set γ which is the endpoints of the segment $[0,1]$, $m_{(8.7)}$ is a sufficiently small, positive number. The estimates (8.6) and (8.7) hold, for example, when

$$U,\ W\in H^{(2n+4+\nu)}(\overline{G}),\quad \nu>0. \tag{8.8}$$

The inclusions (8.8) are guaranteed if $a\in H^{(\alpha+2n-1)}(\overline{G})$, $c,\ p,\ f\in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi\in H^{(\alpha+2n)}(\overline{G})$, $\alpha>4$, $n\ge 0$ and condition (8.1) is fulfilled. We summarize these results in the following theorem.

Theorem 5 *Assume in equation* (2.1) *that* $a,\ b,\ c,\ p,\ f\in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi\in H^{(\alpha+2n)}(\overline{G})$, $\alpha>4$, $n\ge 0$ *and let the condition* (8.1) *be fulfilled. Then, for the solution,* $u(x,t)$, *of problem* (2.1), *and for its components in the representation* (8.4), *it follows that* $u,U,W\in H^{(\alpha+2n)}(\overline{G})$ *and that the estimates* (8.2), (8.6), (8.7) *hold.*

The proof of the theorem is similar to the proof in [10], where the equation

$$\varepsilon\, a(x,t)\,\frac{\partial^2}{\partial x^2}u(x,t)+b(x,t)\,\frac{\partial}{\partial x}u(x,t)-c(x,t)u(x,t)-p(x,t)\,\frac{\partial u}{\partial t}(x,t)=f(x,t)$$

was considered.

8.2 The proof of Theorem 3

Let us show that the function $\delta_{\bar t}\,z(x,t)$, where $z(x,t)=z_{(6.3)}(x,t)$ is the solution of the difference problem (6.3), approximates the function $\delta_{\bar t}\,u(x,t)$ ε-uniformly. For simplicity we assume $a(x,t)$, $b(x,t)$ to be constant on $\overline{G}$. The function $\delta_{\bar t}\,z(x,t)$ is the solution of the difference problem

$$\Lambda_{(8.9)}\delta_{\bar t}\,z(x,t)=f_{(8.9)}(x,t),\qquad (x,t)\in G_h^{[1]}, \tag{8.9a}$$

$$\delta_{\bar t}\,z(x,t)=\varphi_{(8.9)}(x,t),\qquad (x,t)\in S_h^{[1]}. \tag{8.9b}$$

Here

$$\overline{G}_h^{[k]}=\overline{G}_h\cap\{t\ge k\tau\}\,,\quad G_h^{[k]}=G_h\cap\{t>k\tau\}\,,\quad S_h^{[k]}=\overline{G}_h^{[k]}\setminus G_h^{[k]},\quad k\ge 1,$$

$$\Lambda_{(8.9)}\delta_{\bar t}z(x,t)\equiv\{\,\varepsilon\ a\delta_{\overline{x}\widehat{x}}+b\delta_x-\check c(x,t)-p_{\bar t}(x,t)-\check p(x,t)\,\delta_{\bar t}\}\,\delta_{\bar t}z(x,t),$$

$$f_{(8.9)}(x,t)=f_{\bar t}(x,t)+c_{\bar t}(x,t)z(x,t),$$

$$\varphi_{(8.9)}(x,t)=\varphi_{\bar t}(x,t),\quad x=0,\,d,\quad (x,t)\in S_h^{[1]},$$

$$\varphi_{(8.9)}(x,t)=\varphi^0_{(8.9)}(x)\equiv\tau^{-1}\,[\,z(x,\tau)-\varphi(x,0)\,]\,,\quad t=\tau,\quad (x,t)\in S_h^{[1]},$$

$$\check v(x,t)=v(x,t-\tau)\ \textit{where}\ \check v(x,t)\ \textit{is one of the functions}\ \check c(x,t),\ \check p(x,t).$$

The function $\delta_{\bar t}u(x,t)\equiv[u(x,t)-u(x,t-\tau)]/\tau$, $(x,t)\in\overline{G}$, $t\ge\tau$ is the solution of the differential problem

$$L_{(8.10)}\delta_{\bar t}u(x,t)=f_{(8.10)}(x,t),\qquad (x,t)\in G^{[1]}, \tag{8.10a}$$

$$\delta_{\bar t}u(x,t)=\varphi_{(8.10)}(x,t),\qquad (x,t)\in S^{[1]}. \tag{8.10b}$$

Here

$$\overline{G}^{[k]} = \overline{G} \cap \{t \geq k\tau\}, \quad G^{[k]} = G \cap \{t > k\tau\}, \quad S^{[k]} = \overline{G}^{[k]} \setminus G^{[k]}, \quad k \geq 1,$$

$$L_{(8.10)}\delta_{\overline{t}}u(x,t) \equiv \varepsilon a\frac{\partial^2}{\partial x^2} + b\frac{\partial}{\partial x} - \check{c}(x,t) - p_{\overline{t}}(x,t) - \check{p}(x,t)\frac{\partial}{\partial t}\,\delta_{\overline{t}}\,u(x,t),$$

$$f_{(8.10)}(x,t) = f_{\overline{t}}(x,t) + c_{\overline{t}}(x,t)u(x,t) + p_{\overline{t}}(x,t)\left(\frac{\partial u}{\partial t}(x,t) - \delta_{\overline{t}}\,u(x,t)\right),$$

$$\varphi_{(8.10)}(x,t) = \varphi_{\overline{t}}(x,t), \quad x = 0,\, d, \quad (x,t) \in S^{[1]},$$

$$\varphi_{(8.10)}(x,t) = \varphi^0_{(8.10)}(x) \equiv \tau^{-1}\left[\, u(x,\tau) - \varphi(x,0)\,\right], \quad t = \tau, \quad (x,t) \in S^{[1]}.$$

Let us estimate

$$\varphi^0_{(8.10)}(x) - \varphi^0_{(8.9)}(x) = \tau^{-1}\omega(x,\tau),$$

where

$$\omega(x,t) = u(x,t) - z(x,t), \quad (x,t) \in \overline{G}_h.$$

The function $\omega(x,t)$ is the solution of the problem

$$\Lambda_{(6.3)}\,\omega(x,t) = (\Lambda_{(6.3)} - L_{(2.1)})\,u(x,t), \quad (x,t) \in G_h, \quad \omega(x,t) = 0, \quad (x,t) \in S_h.$$

The above assumptions and Theorem 5 lead to the estimates of the truncation error (the deduction technique for these estimates are showed, for example, in [10, 8])

$$\left|\, \left(\Lambda_{(6.3)} - L_{(2.1)}\right)\, U(x,t)\,\right| \leq M\,\left[\,N^{-1}\ln\,N + \tau\,\right], \quad (x,t) \in G_h$$

$$\left|\left(\Lambda_{(6.3)} - L_{(2.1)}\right) W(x,t)\,\right| \leq M\left[\varepsilon^{-1}N^{-1}\ln N\exp(-m\varepsilon^{-1}x) + \tau\right], \quad (x,t) \in G_h, \quad x \leq \sigma,$$

where $U(x,t)$ and $W(x,t)$ are the regular and singular parts of the solution from (8.4); $\sigma = \sigma_{(4.1)}$, $m = m_{(8.7)}$. For the components $W(x,t)$ and $W^h(x,t)$ the following estimate is also satisfied

$$\left|W(x,t)\right|, \left|W^h(x,t)\right| \leq M\,N^{-1}, \quad (x,t) \in \overline{G}_h, \quad x \geq \sigma.$$

Here $W^h(x,t)$ is the solution of the problem

$$\Lambda_{(6.3)}\,W^h(x,t) = 0, \quad (x,t) \in G_h, \quad W^h(x,t) = W\,(x,t) \quad (x,t) \in S_h.$$

Using the maximum principle we estimate $\omega(x,t)$

$$|\,\omega(x,t)\,| \leq M\,\left[\,N^{-1}\ln\,N + \tau\,\right]\,t, \quad (x,t) \in \overline{G}_h.$$

Further, for the derivatives we proceed similarly. On the boundary we have

$$|\,\delta_{\overline{t}}u(x,\tau) - \delta_{\overline{t}}z(x,\tau)\,| = \left|\,\varphi^0_{(8.10)}(x) - \varphi^0_{(8.9)}(x)\,\right| \leq M\,\left[\,N^{-1}\ln\,N + \tau\,\right],$$

$$(x,t) \in S^{[1]}_h, \quad t = \tau,$$

i.e. the function $\delta_{\overline{t}}z(x,\tau)$ approximates $\delta_{\overline{t}}u(x,\tau)$ ε-uniformly. Now, it is easy to see that the solution of the difference problem (8.9) approximates the solution of the differential problem (8.10) for the divided difference. Thus, using the same argument as above, we derive the estimate

$$|\,\delta_{\overline{t}}u(x,t) - \delta_{\overline{t}}z(x,t)\,| \;\leq\; M\,\left[\,N^{-1}\ln\,N + \tau\,\right], \quad (x,t) \in \overline{G}^{[1]}_h.$$

Now, for the 2nd difference derivative we show that under condition (6.6) the function $\delta_{2\bar{t}}\, z(x,t)$ approximates the function $\delta_{2\bar{t}}\, u(x,t)$ ε-uniformly on the set $\overline{G}_h^{[2]}$. So, the functions $\delta_{2\bar{t}} z(x,t)$ and $\delta_{2\bar{t}} u(x,t)$ are solutions of equations

$$\Lambda_{(8.11)} \delta_{2\bar{t}} z(x,t) = f_{(8.11)}(x,t), \quad (x,t) \in G_h^{[2]}, \tag{8.11a}$$

$$L_{(8.12)} \delta_{2\bar{t}} u(x,t) = f_{(8.12)}(x,t), \quad (x,t) \in G_h^{[2]}. \tag{8.12a}$$

The equations are found by applying the operator $\delta_{\bar{t}}$ to the equations (8.9a), (8.10a). At the left and the right boundary the following conditions are satisfied:

$$\delta_{2\bar{t}}\, z(x,t) = \varphi_{(8.11)}(x,t), \quad (x,t) \in S_h^{[2]}, \tag{8.11b}$$

$$\delta_{2\bar{t}} u(x,t) = \varphi_{(8.12)}(x,t), \quad (x,t) \in S_h^{[2]}, \tag{8.12b}$$

where

$$\varphi_{(8.11)}(x,t) = \varphi_{\,2\bar{t}}(x,t), \quad x = 0,\ d, \quad (x,t) \in S_h^{[2]}, \tag{8.11c}$$

$$\varphi_{(8.11)}(x,t) = \varphi^0_{(8.11)}(x) \equiv \delta_{\,2\bar{t}} z_{(6.3)}(x,t), \quad t = 2\tau, \quad (x,t) \in S_h^{[2]},$$

$$\varphi_{(8.12)}(x,t) = \varphi_{\,2\bar{t}}(x,t), \quad x = 0,\ d, \quad (x,t) \in S^{[2]}, \tag{8.12c}$$

$$\varphi_{(8.12)}(x,t) = \varphi^0_{(8.12)}(x) \equiv \delta_{\,2\bar{t}}\ u\ (x,t), \quad t = 2\tau, \quad (x,t) \in S^{[2]}.$$

First we estimate

$$\varphi^0_{(8.12)}(x) - \varphi^0_{(8.11)}(x) = \delta_{2\bar{t}}\, u(x,t) - \delta_{2\bar{t}}\, z(x,t), \quad t = 2\tau.$$

For this purpose we write the function $u(x,t)$ in a Taylor expansion in t

$$u(x,t) = a^{(1)}(x)t + a^{(2)}(x)t^2 + v_2(x,t) \equiv u^{[2]}(x,t) + v_2(x,t), \quad (x,t) \in \overline{G}, \tag{8.13}$$

where the coefficients $a^{(1)}(x)$, $a^{(2)}(x)$ should be determined. Inserting $u(x,t)$, in its form (8.13), into equation (2.1a) we come to the system

$$\begin{aligned} -p(x,0)a^{(1)}(x) &= f(x,0), \\ -2p(x,0)a^{(2)}(x) + \varepsilon a \frac{\partial^2}{\partial x^2} a^{(1)}(x) + b\frac{\partial}{\partial x} a^{(1)}(x) - & \\ -\left(c(x,0) + \frac{\partial}{\partial t} p(x,0) \right) a^{(1)}(x) &= \frac{\partial}{\partial t} f(x,0), \end{aligned}$$

from which the functions $a^{(1)}(x)$, $a^{(2)}(x)$ can be found successively. The function $v_2(x,t)$ is the solution of the boundary value problem

$$\begin{aligned} L_{(2.1)} v_2(x,t) = f_{(8.14)}(x,t) &\equiv f(x,t) - L_{(2.1)} u^{[2]}(x,t), \quad (x,t) \in G, \\ v_2(x,t) = \varphi_{(8.14)}(x,t) &\equiv \varphi(x,t) - u^{[2]}(x,t), \qquad\quad (x,t) \in S. \end{aligned} \tag{8.14}$$

Estimating $f_{(8.14)}(x,t)$ and $\varphi_{(8.14)}(x,t)$, and using the maximum principle we derive the estimate

$$|\, v_2(x,t)\,| \le M\, t^3, \quad (x,t) \in \overline{G}. \tag{8.15}$$

Further we have to construct the function $z(x,t)$ in the form

$$z(x,t) = (b_0^{(1)}(x) + b_1^{(1)}(x)\tau)t + b_0^{(2)}(x)t^2 + v_2^h(x,t) \equiv z^{[2]}(x,t) + v_2^h(x,t), \quad (x,t) \in \overline{G}_h,$$

i.e. as an expansion in powers of τ and t. Inserting $z(x,t)$ into equation (6.3), we arrive at the equations

$$\begin{aligned} -p(x,0)b_0^{(1)}(x) &= f(x,0), \\ -2p(x,0)b_0^{(2)}(x) + \varepsilon a\frac{\partial^2}{\partial x^2}b_0^{(1)}(x) + b\frac{\partial}{\partial x}b_0^{(1)}(x) - & \\ -\left(c(x,0) + \frac{\partial}{\partial t}p(x,0)\right)b_0^{(1)}(x) &= \frac{\partial}{\partial t}f(x,0), \\ b_0^{(2)}(x) + b_1^{(1)}(x) &= 0. \end{aligned}$$

So, we have

$$z^{[2]}(x,t) = u^{[2]}(x,t) + b_1^{(1)}(x)\tau t, \quad (x,t) \in \overline{G}_h. \tag{8.16}$$

The function $v_2^h(x,t)$ is the solution of the discrete boundary value problem

$$\begin{aligned} \Lambda_{(6.3)}v_2^h(x,t) &= f_{(8.17)}(x,t) \equiv f(x,t) - \Lambda_{(6.3)}z^{[2]}(x,t), \ (x,t) \in G_h, \\ v_2^h(x,t) &= \varphi_{(8.17)}(x,t) \equiv \varphi(x,t) - z^{[2]}(x,t), \qquad (x,t) \in S_h. \end{aligned} \tag{8.17}$$

Taking into account estimates of the functions $f_{(8.17)}(x,t)$ and $\varphi_{(8.17)}(x,t)$, we derive the estimate

$$\left| v_2^h(x,t) \right| \le M \left[N^{-1}\ln N + t \right] t^2, \quad (x,t) \in \overline{G}_h. \tag{8.18}$$

By virtue of relations (8.15), (8.16), (8.18) the following inequality is valid:

$$\left| \varphi^0_{(8.12)}(x) - \varphi^0_{(8.11)}(x) \right| = \left| \delta_{2\bar t}u(x,t) - \delta_{2\bar t}z(x,t) \right| \le M \left[N^{-1}\ln N + \tau \right], \tag{8.19}$$
$$(x,t) \in \overline{G}_h, \quad t = 2\tau.$$

We continue by estimating $\delta_{2\bar t}u(x,t) - \delta_{2\bar t}z(x,t)$ for $t > 2\tau$. Note that the functions $\delta_{2\bar t}u(x,t)$ and $\delta_{2\bar t}z(x,t)$ are solutions of differential and difference equations, obtained from equations (2.1) and (6.3) respectively by applying the operator $\delta_{2\bar t}$. Moreover, the difference equation for $\delta_{2\bar t}z(x,t)$ approximates the differential equation for $\delta_{2\bar t}u(x,t)$ ε-uniformly. On the boundary S_h, for $x = 0$ or $x = 1$, we have $\delta_{2\bar t}u(x,t) = \delta_{2\bar t}z(x,t)$. Taking into account estimate (8.19) we find

$$\left| \delta_{2\bar t}u(x,t) - \delta_{2\bar t}z(x,t) \right| \le M \left[N^{-1}\ln N + \tau \right], \quad (x,t) \in \overline{G}_h, \quad t \ge 2\tau. \tag{8.20}$$

So, we come to the estimates

$$\left| \delta_{\bar t}u(x,t) - \delta_{\bar t}z^{(1)}(x,t) \right| \le M \left[N^{-1}\ln N + \tau \right], \quad (x,t) \in \overline{G}_h, \quad t \ge \tau. \tag{8.21}$$

$$\left| \delta_{2\bar t}u(x,t) - \delta_{2\bar t}z^{(1)}(x,t) \right| \le M \left[N^{-1}\ln N + \tau \right], \quad (x,t) \in \overline{G}_h, \quad t \ge 2\tau.$$

$$\left| u(x,t) - z^{(2)}(x,t) \right| \le M \left[N^{-1}\ln N + \tau^2 \right], \quad (x,t) \in \overline{G}_h.$$

This completes the proof.

Now, as a direct consequence of the theorem, we make two remarks to prepare the proof of Theorem 4.

Remark 1 Above we have found (8.20) for $z^{(k)}(x,t)$, $k=1$. In completely the same way we derive this bound for $k=2$, so that we obtain

$$\left|\delta_{2\overline{t}}\,u(x,t)-\delta_{2\overline{t}}\,z^{(k)}(x,t)\right|\le M\left[N^{-1}\ln N+\tau^{k}\right],\quad (x,t)\in\overline{G}_h,\quad t\ge 2\tau,\quad k\le 2. \tag{8.22}$$

Remark 2 Making use of (8.22), similar to the derivation of estimate (8.21), we also find

$$\left|\delta_{3\overline{t}}\,u(x,t)-\delta_{3\overline{t}}\,z^{(1)}(x,t)\right|\le M\left[N^{-2}\ln N+\tau\right],\quad (x,t)\in\overline{G}_h,\quad t\ge 3\tau. \tag{8.23}$$

We briefly indicate the differences with the proof given above for (8.21). To estimate the difference between $\delta_{3\overline{t}}\,u(x,t)$ and $\delta_{3\overline{t}}\,z(x,t)$ for $t=3\tau$ we represent the function $u(x,t)$ (with condition (6.9)) in the form

$$u(x,t)=a^{(2)}(x)t^2+a^{(3)}(x)t^3+v_3(x,t)\equiv u^{[3]}(x,t)+v_3(x,t),\quad (x,t)\in\overline{G},$$

and the function $z(x,t)$ in the form

$$\begin{aligned}z(x,t)&=u^{[3]}(x,t)+(b_1^{(1)}(x)\tau+b_2^{(1)}(x)\tau^2)t+b_1^{(2)}(x)\tau t^2+v_3^h(x,t)\equiv\\&\equiv z^{[3]}(x,t)+v_3^h(x,t),\quad (x,t)\in\overline{G}_h.\end{aligned}$$

The coefficients of these expansions are found using equations (2.1) and (6.3) respectively. For the coefficients we have the system

$$\begin{aligned}-2p(x,0)a^{(2)}(x)&=\frac{\partial}{\partial t}f(x,0),\\ -3p(x,0)a^{(3)}(x)+\varepsilon a\frac{\partial^2}{\partial x^2}a^{(2)}(x)+b\frac{\partial}{\partial x}a^{(2)}(x)-\left(c(x,0)+2\frac{\partial}{\partial t}p(x,0)\right)a^{(2)}(x)&=\frac{1}{2}\frac{\partial^2}{\partial t^2}f(x,0),\\ -b_1^{(1)}(x)+a^{(2)}(x)&=0,\\ -2p(x,0)b_1^{(2)}(x)+\frac{\partial}{\partial t}p(x,0)a^{(2)}(x)+3p(x,0)a^{(3)}(x)-&\\ -\left(\frac{\partial}{\partial t}\,p(x,0)+c(x,0)\right)b_1^{(1)}(x)+\varepsilon a\frac{\partial^2}{\partial x^2}b_1^{(1)}(x)+b\frac{\partial}{\partial x}b_1^{(1)}(x)&=0,\\ -b_2^{(1)}(x)-a^{(3)}(x)+b_1^{(2)}(x)&=0.\end{aligned}$$

The unknown functions $a^{(2)}$, $a^{(3)}$, $b_1^{(1)}$, $b_1^{(2)}$, $b_2^{(1)}$ can be found successively. For the function $v_3(x,t)$ and $v_3^h(x,t)$ the following estimates are derived

$$\begin{aligned}|\,v_3(x,t)\,|&\le Mt^4,\quad (x,t)\in\overline{G},\\ \left|v_3^h(x,t)\right|&\le M\left[N^{-1}\ln N+t\right]t^3,\quad (x,t)\in\overline{G}_h.\end{aligned}$$

From these inequalities and the expression for $z^{[3]}(x,t)$ it follows that (8.23) holds ε-uniformly for $t=3\tau$. The remainder of the proof of the estimate (8.23) repeats with small variations the proof of the estimate (8.21).

8.3 The proof of Theorem 4

Notice that, if the following relations hold for the functions $z^{(1)}(x,t)$, $z^{(2)}(x,t)$

$$\left|\delta_{3\overline{t}}\,u(x,t)-\delta_{3\overline{t}}\,z^{(1)}(x,t)\right|\le M\left[N^{-1}\ln N+\tau\right],\quad (x,t)\in G_h,\quad t\ge 3\tau, \tag{8.24}$$

$$\left|\delta_{2\overline{t}}\,u(x,t)-\delta_{2\overline{t}}\,z^{(2)}(x,t)\right|\le M\left[N^{-1}\ln N+\tau^2\right],\quad (x,t)\in G_h,\quad t\ge 2\tau,$$

then for the difference $u(x,t) - z^{(3)}(x,t) \equiv \omega^{(3)}(x,t)$ we obtain

$$\left| \Lambda_{(6.3)} \omega^{(3)}(x,t) \right| \le M \left[N^{-1} \ln N + \tau^3 \right], \quad (x,t) \in G_h, \quad \omega^{(3)}(x,t) = 0, \quad (x,t) \in S_h.$$

Hence we have

$$\left| u(x,t) - z^{(3)}(x,t) \right| \le M \left[N^{-1} \ln N + \tau^3 \right], \quad (x,t) \in \overline{G}_h.$$

Thus, for the proof of the theorem it is sufficient to show inequalities (8.24). These inequalities follow from (8.22), (8.23). This completes the proof of Theorem 4.